A classification of S^3-bundles over S^4

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ABSTRACT. We classify the total spaces of S^3-bundles over S^4 up to homotopy equivalence, homeomorphism and diffeomorphism. These total spaces have been of interest to both topologists and geometers. They play an important role in the recent work of Grove and Ziller [GZ], where it is shown that each of these total spaces admits metrics with non-negative sectional curvature.

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1. Introduction

For almost fifty years, 3-sphere bundles over the 4-sphere have been of interest to both topologists and geometers. In 1956, Milnor [M] proved that all S^3-bundles over S^4 with Euler class $e = \pm 1$ are homeomorphic to S^7. He also showed that some of these bundles are not diffeomorphic to S^7, he thereby exhibited the first examples of exotic spheres. Shortly thereafter, in 1962, Eells and Kuiper [EK] classified all such bundles with Euler class $e = \pm 1$ up to diffeomorphism and showed that 15 of the 27 seven dimensional exotic spheres can be described as S^3-bundles over S^4.

In 1974 Gromoll and Meyer [GM] constructed a metric with non-negative sectional curvature on one of these sphere bundles, exhibiting the first example of an exotic sphere with non-negative curvature. Very recently Grove and Ziller [GZ] showed that the total space of every S^3-bundle over S^4 admits a metric with non-negative sectional curvature. Motivated by these examples they asked for a classification of these manifolds up to homotopy equivalence, homeomorphism and diffeomorphism (see problem 5.2 in [GZ]). Although a partial classification of this

type is contained in [JW] and [T], it seems that this has not been carried out in the literature in complete generality. It is the purpose of this note to do this.

In the following description we use the notation of Tamura [T]. Let $\mathfrak{M} = \{M, p, S^3, S^4, SO(4)\}$ denote the fibre bundles over the 4-sphere S^4 with total space M, fibre the 3-sphere S^3 and structure group $SO(4)$. Equivalence classes of such bundles are in one-to-one correspondence with $\pi_3(SO(4)) \cong \mathbf{Z} \oplus \mathbf{Z}$. More precisely, the generators ρ and σ of $\pi_3(SO(4))$ are given by

$$\rho(u)\ v = u\ v\ u^{-1}\ , \quad \sigma(u)\ v = u\ v\ ;$$

where u and v denote the quaternions with norm 1, i.e. we have have identified S^3 with the unit quaternions. With this choice of generators each pair of integers (m, n) gives the element $m\ \rho + n\ \sigma \in \pi_3(SO(4))$ and thus determines the bundle

$$\mathfrak{M}_{m,n} = \{M_{m,n}, p, S^3, S^4, SO(4)\}.$$

Remark 1.1. The definition we use is different from the one given by Milnor in [M]. He uses two integers (k, l) corresponding to a different choice of generators of $\pi_3(SO(4)) \cong \mathbf{Z} \oplus \mathbf{Z}$. The two pairs (m, n) and (k, l) are related by $k + l = n$, $l = -m$.

The first level of classification of the manifolds $M_{m,n}$ is up to homotopy equivalence. In a remarkable paper [JW] James and Whitehead succeeded in classifying the manifolds $M_{m,n}$ up to homotopy except for the case $n = 2$. We complete this last case by means of standard tools of homotopy theory which were not available in early 1950s when James and Whitehead wrote their article.

Theorem 1.2. *The manifolds $M_{m,n}$ and $M_{m',n'}$ are homotopy equivalent if and only if $n = n'$ and $m \equiv \pm m' \mod (n, 12)$.*

Note that since $H^4(M_{m,n}; \mathbf{Z}) \cong \mathbf{Z}/n$, necessarily $n = n'$ if $M_{m,n}$ and $M_{m',n'}$ are homotopy equivalent.

The next level, the homeomorphism classification, was studied by Tamura [T], who gave sufficient conditions for two manifolds $M_{m,n}$ and $M_{m',n}$ to be homeomorphic. We use basic results in surgery theory to show that, in fact, Tamura's conditions are necessary as well. Let $p_1(M_{m,n}) \in H^4(M_{m,n}; \mathbf{Z})$ denote the first Pontrjagin class of $M_{m,n}$.

Theorem 1.3. *The manifolds $M_{m,n}$ and $M_{m',n}$ are homeomorphic if and only if*
(a) *$M_{m,n}$ and $M_{m',n}$ are homotopy equivalent and*
(b) *$|p_1(M_{m,n})| = |p_1(M_{m',n})|$, or equivalently $4\ m \equiv \pm 4\ m' \mod n$.*

Remark 1.4. The homeomorphism classification is exactly the same as the *PL*-classification for the manifolds $M_{m,n}$. Indeed, we take advantage of this fact in the proof of Theorem 1.3.

The diffeomorphism classification of the manifolds $M_{m,n}$ is related to the famous article by Milnor [M], where he proved that the manifolds $M_{m,1}$ are exotic homotopy spheres. Eells and Kuiper [EK] gave a diffeomorphism classification of the homotopy spheres $M_{m,1}$ by introducing a new invariant which is known as the μ-invariant. We now recall its definition.

Assume that $M = M^7$ is a closed, smooth *Spin* manifold such that M is a *Spin* boundary, i.e. $M = \partial W^8$, where W^8 is a closed, smooth *Spin* manifold. In

addition, we require W to satisfy the μ-*condition*, see [EK]. The μ-condition allows the pull-back of the Pontrjagin classes of W to $H^*(W, M)$. Then the μ-invariant is defined as

$$(1) \qquad \mu(M^7) \equiv \frac{1}{2^7 \cdot 7} \; \{p_1^2(W) - 4 \; \tau[W]\} \quad \mod 1$$

Here $\tau[W]$ stands for the signature of W. The most important feature of the μ-invariant is that it is an invariant of the diffeomorphism type of M. We use the μ-invariant to prove the following theorem.

Theorem 1.5. *The manifolds $M_{m,n}$ and $M_{m',n}$ are diffeomorphic if and only*
    (a) *$M_{m,n}$ and $M_{m',n}$ are homeomorphic and*
    (b) *$\mu(M_{m,n}) = \mu(M_{m',n})$.*

Smooth surgery theory (see [MM]) implies that there are exactly 28 different smooth manifolds homotopy equivalent to $M_{m,n}$. A natural question to ask is which manifolds homotopy equivalent to $M_{m,n}$ may be represented by manifolds $M_{m',n}$. In a corollary to Theorem 1.5 we give a complete answer to this question.

It is a pleasure to acknowledge very helpful discussions with Jie Wu and his permission to use his thesis [W]. Recently we became aware that D. Crowley from Indiana University is independently working on a project close to the subject of this paper.

2. Completing the homotopy equivalence classification

We first observe that Theorem 1.2 is proved in Theorem 1.7 of [JW] for all cases except for $n = 2$. Thus we now consider only the manifolds $M_{m,2}$.

First we recall some basic facts from [JW]. Let $p : B \longrightarrow S^4$ be a sphere bundle with fibre S^3. Then there is an exact sequence of homotopy groups:

$$\cdots \rightarrow \pi_j(S^3) \xrightarrow{i_*} \pi_j(B) \xrightarrow{p_*} \pi_j(S^4) \xrightarrow{\partial_*} \pi_{j-1}(S^3) \xrightarrow{i_*} \pi_{j-1}(B) \rightarrow \cdots$$

In particular, we have to understand the homomorphism $\partial_* : \pi_4(S^4) \longrightarrow \pi_3(S^3)$ in order to compute the various homology, cohomology and homotopy groups of B. In the case of $B = M_{m,2}$ we know that $H_3(M_{m,2}; \mathbf{Z}) \cong \mathbf{Z}_2$, and by the Hurewicz Theorem $\pi_3(M_{m,2}) \cong \mathbf{Z}_2$. Hence the homomorphism $\pi_4(S^4) \xrightarrow{\partial_*} \pi_3(S^3)$ is multiplication by 2.

We assume that the spheres S^3 and S^4 are oriented by choosing the standard generators $\iota_3 \in \pi_3(S^3)$ and $\iota_4 \in \pi_4(S^4)$. Both generators ι_3, ι_4 are given by the identity maps on the spheres. To give an orientation to the manifold $M_{m,2} = B$ one chooses on orientation on the base S^4 and on the fiber S^3. Here we have that $\partial_*(\iota_4) = 2 \iota_3$. Thus, as pointed out in [JW], the manifolds $M_{m,2}$ have a *preferred* orientation which is defined naturally by the standard orientations of the base and fiber spheres. However, the manifolds $M_{m,2}$ still may have (and indeed have) a self-homotopy map $f : M_{m,2} \longrightarrow M_{m,2}$ of degree -1.

James & Whitehead [JW] give the following definitions. Two manifolds $M_{m,2}$ and $M_{m',2}$ are called *opposed to each other* if there exists a homotopy equivalence $f : M_{m,2} \longrightarrow M_{m',2}$ of degree -1. A manifold $M_{m,2}$ is called *self-opposed* if there exists a self-homotopy equivalence $f : M_{m,2} \longrightarrow M_{m,2}$ of degree -1. As it is emphasized in [JW, p. 151], there is no ambiguity here if $n > 2$ or $n = 1$ since then every homotopy equivalence $f : M_{m,n} \longrightarrow M_{m',n}$ must be of degree $+1$ and this

fact is used in [JW] to obtain the homotopy equivalence classification for $n > 2$ and $n = 1$. Hence in order to complete the classification for $n = 2$ we need to show that up to homotopy every homotopy equivalence $f : M_{m,2} \longrightarrow M_{m',2}$ must be of degree $+1$. We prove this fact by showing that every manifold $M_{m,2}$ is self-opposed.

Given a manifold $M_{m,2} = X$ we construct a self-homotopy map $f : M_{m,2} \longrightarrow M_{m,2}$ of degree -1 by considering X as a CW-complex. The cell-structure is described in [JW]. In fact, the 4-skeleton $X^{(4)}$ of X is the Moore space, i.e. $X^{(4)} = \Sigma^2 \mathbf{RP}^2$, and the space X is obtained from $X^{(4)}$ by attaching a 7-cell e^7.

Let $\phi : S^6 \longrightarrow \Sigma^2 \mathbf{RP}^2$ be an attaching map for the cell e^7 which represents some element $[\phi] \in \pi_6(\Sigma^2 \mathbf{RP}^2)$. Consider the long exact sequence of the pair $(\Sigma^2 \mathbf{RP}^2, S^3)$:

$$\cdots \to \pi_6(S^3) \xrightarrow{i_*} \pi_6(\Sigma^2 \mathbf{RP}^2) \xrightarrow{j_*} \pi_6(\Sigma^2 \mathbf{RP}^2, S^3) \xrightarrow{\partial} \pi_5(S^3) \to \cdots$$

The above groups and homomorphisms are amongst those computed by Jie Wu in his thesis [W]. Using his results we obtain the following commutative diagram.

$$(2) \quad
\begin{array}{ccccccccc}
\cdots \to \pi_6(S^3) & \xrightarrow{i_*} & \pi_6(\Sigma^2 \mathbf{RP}^2) & \xrightarrow{j_*} & \pi_6(\Sigma^2 \mathbf{RP}^2, S^3) & \xrightarrow{\partial_*} & \pi_5(S^3) \to \cdots \\
\big\downarrow{\cong} & & \big\downarrow{\cong} & & \big\downarrow{\cong} & & \big\downarrow{\cong} \\
\cdots \to \mathbf{Z}_4(\nu') & \xrightarrow{i_*} & \mathbf{Z}_4(\lambda) \oplus \mathbf{Z}_2(\omega) & \xrightarrow{j_*} & \mathbf{Z}_2(\lambda'') \oplus \mathbf{Z}_2(\omega'') & \xrightarrow{\partial_*} & \mathbf{Z}_2(b) \to \cdots
\end{array}$$

Here $\mathbf{Z}_4(\nu')$ indicates that ν' is a generator of $\mathbf{Z}_4$ and $i_*(\nu') = (2\lambda, 0)$, $j_*(\lambda, 0) = (\lambda'', 0)$, and $j_*(0, \omega) = (0, \omega'')$.

In particular, the homomorphism $\partial_* : \pi_6(\Sigma^2 \mathbf{RP}^2, S^3) \to \pi_5(S^3)$ is trivial. Let $\kappa \in \pi_4(\Sigma^2 \mathbf{RP}^2, S^3)$ be the element given by the characteristic map $(D^4, S^3) \to (\Sigma^2 \mathbf{RP}^2, S^3)$, and $\iota_3 \in \pi_3(S^3)$ be the standard generator of $\pi_3(S^3)$. James & Whitehead [JW, Section 5] show that $j_*([\phi]) = [\kappa, \iota_3] \in \pi_6(\Sigma^2 \mathbf{RP}^2, S^3)$ where $\phi : S^6 \to \Sigma^2 \mathbf{RP}^2$ was an attaching map for the cell e^7.

Remark 2.1. It is emphasized in [JW, Theorem 1.4.] that a necessary condition for a manifold $M_{m,2}$ to be self-opposed is that *the Whitehead product* $[\kappa, \iota_3] \in \pi_6(\Sigma^2 \mathbf{RP}^2, S^3)$ *be of order* 2. This condition clearly holds since $\pi_6(\Sigma^2 \mathbf{RP}^2, S^3) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. There is also a sufficient condition given in [JW, Theorem 1.4.]. Namely, *if* $2[\kappa, \iota_3] = 0$ *in the group* $\pi_6(\Sigma^2 \mathbf{RP}^2, S^3)$ *and* $\pi_7(S^4)$ *is infinite cyclic, then the manifolds* $M_{m,2}$ *are self-opposed.* This condition may be a source for confusion because the group $\pi_7(S^4)$ is not infinite cyclic, but rather $\pi_7(S^4) \cong \mathbf{Z} \oplus \mathbf{Z}_{12}$, see [R].

However, James & Whitehead also provide a true necessary and sufficient condition for a manifold $M_{m,2}$ to be self-opposed in [JW, Lemma 4.1.]. Let $\phi : S^6 \to \Sigma^2 \mathbf{RP}^2$ be an attaching map for the top cell of $M_{m,2}$. Then the following holds (see [JW, Lemma 4.1]).

Fact. $(\star)$ *A manifold* $M_{m,2}$ *is self-opposed if and only if* $2[\phi] = i_*([\eta, \iota_3])$, *where* $\eta \in \pi_4(S^3) \cong \mathbf{Z}_2$ *is the generator, and* $\iota_3 \in \pi_3(S^3)$ *is the standard generator (as above).*

Note that $[\eta, \iota_3] = \nu'$ in our notation, see for example [To], and that $i_*(\nu') = (2\lambda, 0)$. Thus, to complete the classification, it is enough to prove the following

Lemma 2.2. *Let $\phi : S^6 \longrightarrow \Sigma^2 \mathbf{RP}^2$ be an attaching map for the top cell of $M_{m,2}$. Then $2[\phi] = i_*(\nu')$.*

Proof. First we show that the Whitehead product $[\kappa, \iota_3] \in \pi_6(\Sigma^2 \mathbf{RP}^2, S^3)$ is non-zero. Assume to the contrary that $[\kappa, \iota_3] = 0$. As $[\kappa, \iota_3] = j_*([\phi])$, the attaching map $\phi : S^6 \longrightarrow \Sigma^2 \mathbf{RP}^2$ may then be lifted to a map $\tilde{\phi} : S^6 \longrightarrow S^3$. This leads to a contradiction as follows.

If $[\tilde{\phi}] = 0$ in $\pi_6(S^3)$, then $M_{m,2} \cong \Sigma^2 \mathbf{RP}^2 \vee S^7$, and we have a long exact sequence in homotopy:

$$\cdots \to \pi_k(S^3) \longrightarrow \pi_k(\Sigma^2 \mathbf{RP}^2 \vee S^7) \longrightarrow \pi_k(S^4) \longrightarrow \pi_{k-1}(S^3) \to \cdots$$

However, $\pi_6(S^3) = \mathbf{Z}_2$, $\pi_6(\Sigma^2 \mathbf{RP}^2 \vee S^7) = \mathbf{Z}_4 \oplus \mathbf{Z}_2$, $\pi_6(S^4) = \mathbf{Z}_2$, and exactness of the above sequence fails.

If $[\tilde{\phi}] \neq 0$ in $\pi_6(S^3)$ then $[\tilde{\phi}] = \nu'$. Let $Y = S^3 \cup_{\tilde{\phi}} D^7$. Then we obtain a map $g : Y \to M_{m,2}$ which on homology induces an epimorphism

$$g_*^{(3)} : \mathbf{Z} \cong H_3(Y; \mathbf{Z}) \to H_3(M_{m,2}; \mathbf{Z}) \cong \mathbf{Z}_2,$$

and an isomorphism

$$g_*^{(7)} : \mathbf{Z} \cong H_7(Y; \mathbf{Z}) \to H_7(M_{m,2}; \mathbf{Z}) \cong \mathbf{Z}.$$

Let $\alpha' \in H_3(Y; \mathbf{Z})$, $\sigma' \in H_7(Y; \mathbf{Z})$, $\beta \in H^4(M_{m,2}; \mathbf{Z})$, be generators. Using the naturality of the cap product and Poincaré duality we obtain that

$$g_*^{(7)}(\sigma') \cap \beta = g_*^{(3)}(\alpha') = g_*^{(3)}(\sigma' \cap g_{(4)}^*(\beta)),$$

hence $\alpha' = \sigma' \cap g_{(4)}^*(\beta)$ which fails since $g_{(4)}^*(\beta) = 0$. Therefore we obtain a contradiction to the existence of the map $\tilde{\phi} : S^6 \to S^3$ which implies that $j_*([\phi]) = [\kappa, \iota_3] \neq 0$.

Thus the element $[\phi] \in \pi_6(\Sigma^2 \mathbf{RP}^2)$ must be $(\pm\lambda, 0)$ or $(\pm\lambda, \omega)$ In both cases $2[\phi] = (\pm 2\lambda, 0) = (2\lambda, 0) = i_*([\eta, \iota_3]) = i_*(\nu')$. This proves Lemma 2.2. $\qquad\square$

Remark 2.3. In fact, it is not hard to prove that $[\phi] = (\pm\lambda, 0) \in \pi_6(\Sigma^2 \mathbf{RP}^2)$. To do this, it is enough to observe that the the pinching map $\rho : \Sigma^2 \mathbf{RP}^2 \to S^4$ induces a map in homotopy $\rho_* : \pi_6(\Sigma^2 \mathbf{RP}^2) \to \pi_6(S^4)$ which takes the element $[\phi]$ to zero.

3. **Homeomorphism classification**

It is easy to compute the cohomology of $M_{m,n}$ with integer coefficients by using the Serre spectral sequence:

$$H^0(M_{m,n}) \cong H^7(M_{m,n}) \cong \mathbf{Z}$$
$$H^4(M_{m,n}) \cong \mathbf{Z}_n$$
$$H^i(M_{m,n}) \cong 0 \quad \text{for all} \quad i \neq 0, 4, 7.$$

Computing the Pontrjagin classes [T] of the bundles $\mathfrak{M}_{m,n}$ and the total spaces $M_{m,n}$ yields

$$p_1(\mathfrak{M}_{m,n}) = \pm 2\,(2m+n)\,\alpha_4 \in \mathbf{Z}$$
$$p_1(M_{m,n}) = \pm 4\,m\,\beta_4 \in \mathbf{Z}_n$$

where α_4 and β_4 are generators of $H^4(S^4) \cong \mathbf{Z}$ and $H^4(M_{m,n}) \cong \mathbf{Z}_n$, respectively. We now start with a given homotopy type, i.e. we with an equivalence class of manifolds, all homotopy equivalent to each other. By Theorem 1.2 we know that

these manifolds have indices satisfying a certain congruence. In order to use surgery theory to classify these manifolds up to homeomorphism we recall the following definitions based on the work of Sullivan, see [MM] for a detailed description. Let X be a Cat-manifold, where Cat stands for the piecewise linear (PL), topological (Top), or smooth (O) category. Then $\mathfrak{S}_{\mathrm{Cat}}(X)$, the Cat-structure set of X, consists of pairs (L, f) where L is a Cat-manifold and $f : L \to X$ is a simple homotopy equivalence. Two objects (L_1, f_1) and (L_2, f_2) represent the same element in $\mathfrak{S}_{\mathrm{Cat}}(X)$ if there exist an h-cobordism W with $\partial W = L_1 \cup L_2$ and a homotopy equivalence $F : W \to X$. Moreover we denote by $NM_{\mathrm{Cat}}(X)$ the set of equivalence classes of normal cobordisms. Here two normal maps (or surgery problems)

$$
\begin{array}{ccc}
\nu_{L_1} & \xrightarrow{\ \hat{f}_1\ } & \nu_1 \\
\downarrow & & \downarrow \\
L_1 & \xrightarrow{\ f_1\ } & X
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\nu_{L_2} & \xrightarrow{\ \hat{f}_2\ } & \nu_2 \\
\downarrow & & \downarrow \\
L_2 & \xrightarrow{\ f_2\ } & X
\end{array}
$$

are normally cobordant if there exists a cobordism W with $\partial W = L_1 \cup L_2$ and

$$
\begin{array}{ccc}
\nu_W & \xrightarrow{\ \hat{F}\ } & \nu \times I \\
\downarrow & & \downarrow \\
W & \xrightarrow{\ F\ } & X \times I
\end{array}
$$

such that $F|L_1 = f_1$, $\hat{F}|\nu_{L_1} = \hat{f}_1$, $F|L_2 = f_2$, $\hat{F}|\nu_{L_2} = b \circ \hat{f}_2$ where $b : \nu_2 \to \nu_1$ is a bundle isomorphism. Let

$$
P_n = \begin{cases}
\mathbf{Z}, & \text{if } n = 4\,k; \\
\mathbf{Z}_2, & \text{if } n = 4\,k + 2; \\
0, & \text{otherwise.}
\end{cases}
$$

The following two facts are well-known, see [MM].

(1) $NM_{Cat}(X) \cong [X; G/\mathrm{Cat}]$.

(2) *There is an exact sequence of sets, the structure sequence:*

$$
\cdots \longrightarrow NM_{\mathrm{Cat}}(X^n \times I, \partial) \xrightarrow{\ s\ } P_{n+1} \longrightarrow \mathfrak{S}_{\mathrm{Cat}}(X^n) \longrightarrow NM_{\mathrm{Cat}}(X^n) \longrightarrow P_n.
$$

In our case $n = 7$ and the map $NM_{\mathrm{Cat}}(M^7 \times I, \partial) \xrightarrow{\ s\ } P_8$ is an epimorphism in the Top and PL category. Hence (1) and (2) imply that

$$
\mathfrak{S}_{\mathrm{Top}}(M^7_{m,n}) \cong NM_{\mathrm{Top}}(M^7_{m,n}) \cong [M^7_{m,n}; G/\mathrm{Top}].
$$

We now use the structure of the well-known space G/Top. We proceed by first localizing at the prime $p = 2$; G/Top localized at $p = 2$ is just a product of Eilenberg-Maclane spaces.

$$
G/\mathrm{Top}[2] = \prod_k K(\mathbf{Z}_{(2)}, 4k) \times K(\mathbf{Z}_2, 4k - 2).
$$

As there is no torsion in this case we obtain for the homology of $M_{m,n}$ with $\mathbf{Z}_{(2)}$ coefficients

$$
H_q(M_{m,n}; \mathbf{Z}_{(2)}) \cong H_q(M_{m,n}; \mathbf{Z}) \otimes \mathbf{Z}_{(2)}.
$$

Now for the prime $p = 2$ we derive

$$[M_{m,n}\,;G/\mathrm{Top}[2]\,] \quad \cong \quad [M_{m,n}\,,\,\textstyle\prod_k K(\mathbf{Z}_{(2)},4k) \times K(\mathbf{Z}_2,4k-2)]$$

$$\cong \quad H^4(M_{m,n}\,;\mathbf{Z}_{(2)}) \cong \mathbf{Z}_n \otimes \mathbf{Z}_{(2)}\,.$$

Lemma 3.1. $[M_{m,n}^7\,;G/\mathrm{Top}[2]\,] \cong [M_{m,n}^7\,;G/\mathrm{PL}[2]\,]$

Proof. Consider the natural map $j\,:\,G/\mathrm{PL}[2] \longrightarrow G/\mathrm{Top}[2]$ and let F be the homotopy fibre of the map j. The space F is a two-stage Postnikov system [MM, Theorem 4.8].

$$\begin{array}{ccc}
K(\mathbf{Z}_{(2)},4) & \longrightarrow & K(\mathbf{Z}_{(2)},4) \\
\downarrow & & \downarrow \\
F & \longrightarrow & * \\
\downarrow & & \downarrow \\
K(\mathbf{Z}_2,2) & \xrightarrow{\;\beta s_q^2\;} & K(\mathbf{Z}_{(2)},5)
\end{array}$$

As $H^2(M_{m,n}^7\,;\mathbf{Z}_2) = 0$, we can lift maps to $K(\mathbf{Z}_{(2)},4)$ and obtain

$$[M_{m,n}^7\,;F] \cong [M_{m,n}^7\,;K(\mathbf{Z}_{(2)},4)]\,.$$

Thus $[M_{m,n}^7\,;G/\mathrm{PL}[2]\,] \cong H^4(M_{m,n}^7\,;\mathbf{Z}_{(2)}) \cong [M_{m,n}^7\,;G/\mathrm{Top}[2]\,]$. $\square$

The next step is to localize at primes $p \neq 2$, i.e. to study $[M_{m,n}^7\,;G/\mathrm{PL}[1/2]\,]$. By Sullivan's Theorem [MM, Theorem 4.28] there is a natural homotopy equivalence $\sigma : G/\mathrm{PL}[1/2] \longrightarrow \mathrm{BO}[1/2]\,]$. Hence we obtain that

$$[M_{m,n}^7\,;G/\mathrm{PL}[1/2]\,] \cong [M_{m,n}^7\,;\mathrm{BO}[1/2]\,] \cong \widetilde{\mathrm{KO}}^0(M_{m,n}^7) \otimes \mathbf{Z}[1/2]\,.$$

We use the Atiyah-Hirzebruch spectral sequence to compute $\widetilde{\mathrm{KO}}^0(M_{m,n}^7)\otimes\mathbf{Z}[1/2] \cong \mathbf{Z}_n \otimes \mathbf{Z}[1/2]$. We combine the previous calculations in a Cartesian square.

$$\begin{array}{ccc}
[M_{m,n}^7\,;G/\mathrm{PL}] & \longrightarrow & [M_{m,n}^7\,;G/\mathrm{PL}[2]\,] \\
\downarrow & & \downarrow \\
[M_{m,n}^7\,;G/\mathrm{PL}[1/2]\,] & \longrightarrow & [M_{m,n}^7\,;\textstyle\prod_{j\geq 1} K(\mathbf{Q},4j)]
\end{array}$$

As $H^4(M_{m,n}^7\,;\mathbf{Z}) \cong \mathbf{Z}_n$, we obtain

$$[M_{m,n}^7\,;G/\mathrm{PL}] \quad \cong \quad [M_{m,n}^7\,;G/\mathrm{PL}[2]\,] \oplus [M_{m,n}^7\,;G/\mathrm{PL}[1/2]\,]$$

$$\cong \quad (\mathbf{Z}_n \otimes \mathbf{Z}_{(2)}) \oplus (\mathbf{Z}_n \otimes \mathbf{Z}[1/2])$$

$$\cong \quad \mathbf{Z}_n \cong H^4(M_{m,n}^7;\mathbf{Z})\,.$$

Conclusion: The previous arguments imply that

$$\mathfrak{S}_{\mathrm{Top}}(M_{m,n}^7) \cong NM_{\mathrm{Top}}(M_{m,n}^7) \cong [M_{m,n}^7\,;G/\mathrm{Top}] \cong [M_{m,n}^7\,;G/\mathrm{PL}] \cong \mathbf{Z}_n\,.$$

Note that the structure set $\mathfrak{S}_{\mathrm{Top}}(M_{m,n})$ in general may contain manifolds that are not sphere bundles over spheres. Hence we now define

$$\widehat{\mathfrak{S}}_{\mathrm{Top}}(M_{m,n}) \quad = \quad \{(L,f) \in \mathfrak{S}_{\mathrm{Top}}(M_{m,n})|f^*(p_1(M_{m,n})) - p_1(L) \equiv 0 \quad \mathrm{mod}\ 4\}$$

where $f^*(p_1(M_{m,n})) - p_1(L) \in H^4(L; \mathbf{Z}) \cong \mathbf{Z}_n$. This subset of $\mathfrak{S}_{\mathrm{Top}}(M_{m,n})$ has the following properties.

Lemma 3.2. *Let* $n = 2^r a$ *where* $r, a \in \mathbf{Z}$, $r \geq 0$ *and* $(2, a) = 1$.

(a) $\widehat{\mathfrak{S}}_{\mathrm{Top}}(M_{m,n}) \cong \mathbf{Z}_{2^{r-2}a}$ *if* $r \geq 3, a \geq 1$; $\widehat{\mathfrak{S}}_{\mathrm{Top}}(M_{m,n}) \cong \mathbf{Z}_{2^r a}$ *if* $r < 3, a > 1$;

(b) *Every element in* $\widehat{\mathfrak{S}}_{\mathrm{Top}}(M_{m,n})$ *can be represented by some manifold* $M_{m',n}$.

Proof. Part (a) follows from a simple counting argument as every fourth element in $\mathbf{Z}_{2^r a}$ is identified. Statement (b) is a consequence of part (a) as there are enough manifolds $M_{m',n}$ to generate $\mathbf{Z}_{2^{r-2}a}$ or $\mathbf{Z}_{2^r a}$ respectively. If $M_{m',n}$ is homotopy equivalent to $M_{m,n}$, then $m \equiv \pm m' \mod (n, 12)$. But we also have that $p_1(M_{m',n}) \equiv \pm 4\, m' \in \mathbf{Z}_{2^r a}$ which implies that $m' \in \mathbf{Z}_{2^{r-2}a}$ if $r \geq 3$ and $m' \in \mathbf{Z}_{2^r a}$ if $r < 3$. $\qquad\square$

Remark 3.3. The special cases of $r \leq 3$ and $a = 1$ can be treated as follows.

If $a = 1$ and $r = 0$ then $n = 1$ and we discuss this case in Section 4.
If $a = 1$ and $r = 1$ then $n = 2$ and $p_1(M_{m,2}) \equiv 0 \in \mathbf{Z}_2$ and

$$\mathfrak{S}^{(2)}_{\mathrm{Top}}(M_{m,2}) \cong [M_{m,2}, G/\mathrm{Top}]_{(2)} \cong H^4(M_{m,2}; \mathbf{Z}_2) \cong \mathbf{Z}_2.$$

Hence there are two homeomorphism classes of manifolds, which the first Pontrjagin class cannot distinguish as it is always zero. However, by the work of Tamura [T] we know that if $m \equiv \pm m' \mod 2$, then $M_{m,2}$ is homeomorphic to $M_{m',2}$. On the other hand, if $M_{m,2}$ is homeomorphic to $M_{m',2}$ then they are homotopy equivalent and Theorem 1.2 implies $m \equiv \pm m' \mod 2$.

If $a = 1$ and $r = 2$ then $n = 4$ and $p_1(M_{m,4}) = \pm 4\, m \equiv 0 \in \mathbf{Z}_4$. But in this case we obtain from Theorem 1.2 that $M_{m,4}$ is homotopy equivalent to $M_{m',4}$ if and only if $m \equiv \pm m' \mod 4$. Again the work of Tamura [T] implies that if $m \equiv \pm m' \mod 4$, then $M_{m,4}$ is homeomorphic to $M_{m',4}$. Hence for this special case we obtain that $M_{m,4}$ is homotopy equivalent to $M_{m',4}$ if and only if $M_{m,4}$ is homeomorphic to $M_{m',4}$ which is the case if and only if $m \equiv \pm m' \mod 4$. Again these manifolds are not distinguished by their first Pontrjagin class.

As we are only interested in the subset $\widehat{\mathfrak{S}}_{\mathrm{Top}}(M_{m,n})$ for the homeomorphism classification, Theorem 1.3 follows.

4. Diffeomorphism classification

The work of Eells-Kuiper [EK] implies that the μ-invariant of equation (1) determines the differentiable structure of the total spaces $M_{m,n}$. Here $\mu(M_{m,n}) \equiv \mu(W_{m,n}, M_{m,n}) \mod 1$ is computed for any spin coboundary $W_{m,n}$ which satisfies condition μ. Hence in order to complete the classification we need to find a suitable coboundary $W_{m,n}$ and compute the corresponding μ-invariant. The natural choice for $W_{m,n}$ is the associated disk bundle $W_{m,n} \longrightarrow S^4$, $\partial W_{m,n} = M_{m,n}$. This coboundary was already used in [M] and [T]. As $W_{m,n}$ is a disk bundle over S^4 one obtains for the first Pontrjagin class $p_1(W_{m,n}) = \pm 2\,(n + 2m)$ and for the signature $\tau(W_{m,n}) = 1$. Hence we obtain for the μ-invariant of the total spaces $M_{m,n}$

(3)
$$\mu(M_{m,n}) \equiv \tfrac{1}{2^7 \cdot 7}\left\{ p_1^2(W_{m,n}) - 4\,\tau[W_{m,n}] \right\} \mod 1$$

$$\equiv \tfrac{1}{2^5 \cdot 7}\left\{ (n + 2m)^2 - 1 \right\} \mod 1.$$

As shown in [EK], the μ-invariant distinguishes the diffeomorphism type in this case and Theorem 1.5 follows. In order to see exactly how many classes of each homotopy type are represented by smooth manifolds of type $M_{m,n}$ we introduce the following integer valued function.

Definition 4.1. Let $n = 2^r a$ where $a, r \in \mathbf{Z}, r \geq 0$ and $(2, a) = 1$. Then we define the function $d : \mathbf{Z} \times \mathbf{Z} \longrightarrow \mathbf{Z}$ as follows.

  (I) $(3, a) = 1, (7, a) = 1$
    $d(m, n) = 4$ for $[r = 1, m$ even $]$ or $[r \geq 2]$;
    $d(m, n) = 8$ for $[r = 1, m$ odd $]$;
    $d(m, n) = 16$ for $[r = 0]$;
  (II) $(3, a) \neq 1, (7, a) = 1$
    $d(m, n) = 4$ for $[r = 1, m$ even $]$ or $[r = 2, m$ odd $]$ or $[r \geq 3]$;
    $d(m, n) = 8$ for $[r = 1, m$ odd $]$ or $[r = 2, m$ even $]$;
    $d(m, n) = 16$ for $[r = 0]$;
  (III) $(3, a) = 1, (7, a) \neq 1$
    $d(m, n) = 1$ for $[r = 1, m$ even $]$ or $[r \geq 2]$;
    $d(m, n) = 2$ for $[r = 1, m$ odd $]$;
    $d(m, n) = 4$ for $[r = 0]$;
  (IV) $(3, a) \neq 1, (7, a) \neq 1$
    $d(m, n) = 1$ for $[r = 1, m$ even $]$ or $[r = 2, m$ odd $]$ or $[r \geq 3]$;
    $d(m, n) = 2$ for $[r = 1, m$ odd $]$ or $[r = 2, m$ even $]$;
    $d(m, n) = 4$ for $[r = 0]$;

As a consequence of Theorem 1.5 we obtain the following corollary.

Corollary 4.2. *Each homeomorphism type of the manifold $M_{m,n}$ contains exactly $d(m, n)$ distinct classes of smooth manifolds represented by S^3-bundles over S^4.*

Remark 4.3. Note that $d(m, 1) = 16$ for all m shows that there are 16 distinct classes of smooth manifolds $M_{m,1}$. But manifolds of type $M_{m,1}$ are homeomorphic to S^7 for all m, hence we obtain the usual differentiable structure on S^7 together with the 15 exotic spheres that can be described as S^3- bundles over S^4.

Proof of Corollary 4.2. In each case we first combine the homotopy equivalence condition, $m \equiv \pm m' \mod (n, 12)$, and the homeomorphism condition, $4 m \equiv \pm 4 m' \mod n$, to obtain a restriction on m'. We then compute $\mu(M_{m,n}) - \mu(M_{m',n})$ using this condition on m' and calculate the number of values of $\mu(M_{m,n}) - \mu(M_{m',n})$. This calculation reduces to counting the number of squares q^2 modulo $7, 14$ or 56 or counting the number of consecutive products $q(q+1)$ modulo $7, 14$ or 56.

(I) $(3, a) = 1$

   Let $r \geq 4$. Here the homeomorphism condition, $m \equiv \pm m' \mod 2^{r-2}a$, is stronger than the homotopy equivalence condition, $m \equiv \pm m' \mod 4$. We first set $m' = q a 2^{r-2} + m$ for some $q \in \mathbf{Z}$. Then by (3)

$$\mu(M_{m,2^r a}) - \mu(M_{qa2^{r-2}+m,2^r a}) = \frac{1}{2^5 7} [(2 m + 2^r a)^2 - (2 m + 2^{r-1} a q + 2^r a)^2]$$

$$(4) \qquad\qquad\qquad = -\frac{2^{r-4}}{7} a q (m + 2^{r-1} a + 2^{r-3} a q)$$

Recall that $\mu(M_{m,n}) - \mu(M_{m',n}) \in \mathbf{Q}/\mathbf{Z}$, hence we need to calculate the number of values of $a\,q\,(m + 2^{r-1}\,a + 2^{r-3}\,a\,q)$ modulo 7, i.e. $\#\,[a\,q\,(m + 2^{r-1}\,a + 2^{r-3}\,a\,q)]$ mod 7.

If $(7, a) \neq 1$ we obtain one possible value and part of the first case of (III) in Corollary 4.2 follows.

If $(7, a) = 1$, we obtain that

$$\#\,[a\,q\,(m + 2^{r-1}\,a + 2^{r-3}\,a\,q)] \quad \mathrm{mod}\ 7 = \#\,[q\,(2^{r-3}\,a\,q + m + 2^{r-1}\,a)] \quad \mathrm{mod}\ 7,$$

as a is fixed. Hence we are interested in the number of $q\,[q + s\,(m + 2^{r-1}\,a)]$ where s is the inverse of $2^{r-3}\,a$ modulo 7. As the inverse of 2 exists modulo 7, it follows that $m + 2^{r-1}\,a$ is always even modulo 7. Hence $q^2 + q\,s\,(m + 2^{r-1}\,a) \equiv q^2 + 2\,t\,q$ mod $7 \equiv (q + t)^2 - t^2$ mod 7 where $s\,(m + 2^{r-1}\,a) \equiv 2\,t$ mod 7. We now count the number of squares modulo 7 and obtain four possible values.

The case of $m' = q\,a\,2^{r-2} - m$ reduces to exactly the same calculation as

$$\mu(M_{m,2^r a}) - \mu(M_{qa2^{r-2}-m,2^r a}) \;=\; \frac{1}{2^5 7}\,[(2\,m + 2^r\,a)^2 - (-2\,m + 2^{r-1}\,a\,q + 2^r\,a)^2]$$

$$(5) \hspace{4cm} = \;\frac{2^{r-4}}{7}\,a\,(4 + q)\,(m - 2^{r-3}\,a\,q)$$

and we can get from (4) to (5) by the translation $q \mapsto -(4 + q)$.

For $r = 3$ the homotopy equivalence and homeomorphism conditions reduce to $m \equiv \pm\,m'$ mod $4a$ and we first set $m' = 4\,q\,a + m$. Then

$$\mu(M_{m,8a}) - \mu(M_{4qa+m,8a}) \;=\; \frac{1}{2^5 7}\,[(2\,m + 8\,a)^2 - (2\,m + 8\,a\,q + 8\,a)^2]$$

$$(6) \hspace{4cm} = \; -\frac{1}{7}\,a\,q\,(m + 2\,a\,q + 4\,a)$$

If $(7, a) \neq 1$ we obtain one possible value and the first case of (III) in Corollary 4.2 follows.

If $(7, a) = 1$, then $\#\,[a\,q\,(m + 2\,a\,q + 4\,a)]$ mod $7 = \#\,[q\,(2\,a\,q + m + 4\,a)]$ mod $7 = \#\,[q\,(q + 4\,a^{-1}\,m + 2)]$ mod $7 = \#\,[q\,(q + 2\,(2\,a^{-1}\,m + 1))]$ mod 7. As in the previous case the calculation reduces to the number of squares modulo 7 and we obtain four possible values. This completes part of the first case of (I) in Corollary 4.2.

For $m' = 4\,q\,a - m$ the values are unchanged as

$$\mu(M_{m,8a}) - \mu(M_{4qa-m,8a}) \;=\; \frac{1}{2^5 7}\,[(2\,m + 8\,a)^2 - (-2\,m + 4\,a\,q + 8\,a)^2]$$

$$(7) \hspace{4cm} = \; \frac{1}{7}\,a\,(2 + q)\,(m - 2\,a\,q)$$

and we can get from (6) to (7) by the translation $q \mapsto -(2 + q)$.

For $r = 2$ the conditions reduce to $m \equiv \pm\,m'$ mod $4a$ and as before we first set $m' = 4\,q\,a + m$. Then

$$(8) \hspace{2cm} \mu(M_{m,4a}) - \mu(M_{4qa+m,4a}) = -\frac{1}{7}\,a\,q\,(m + 2\,a\,q + 2\,a)$$

Again if $(7, a) \neq 1$ we obtain one possible value modulo 7.

If $(7, a) = 1$, we argue exactly as in the first calculation to obtain the number of squares modulo 7.

For the case of $m' = 4\,q\,a - m$ the values do not change as

$$(9) \qquad \mu(M_{m,4a}) - \mu(M_{4qa-m,4a}) = \frac{1}{7}\,a\,(1+q)\,(m - 2\,a\,q)$$

and we can get from (8) to (9) by the translation $q \mapsto -(1+q)$.

For $r = 1$ the conditions become $m \equiv \pm\, m' \mod 2a$ and hence we first set $m' = 2\,q\,a + m$. Substitution yields

$$(10) \qquad \mu(M_{m,2a}) - \mu(M_{2qa+m,2a}) = -\frac{1}{14}\,a\,q\,(m + a\,q + a)$$

If $(7, a) \neq 1$, then $a = 7\,b$ for some $b \in \mathbf{Z}$ and $\mu(M_{m,2a}) - \mu(M_{2qa+m,2a}) = -\frac{1}{2}\,q\,(m + 7\,b\,q + 7\,b)$. Now $\#\,[q\,(m + 7\,b\,q + 7\,b)] \mod 2 = \#\,[q\,(q + 1 + b^{-1}\,m)] \mod 2 = \#\,[q\,(q + 1 + m)] \mod 2$. If m is even, we obtain the number of consecutive products $q\,(q + 1)$ modulo 2 and if m is odd, the calculation reduces to counting the number of squares modulo 2. This implies that $d(m, n) = 1$ if m is even with $(7, a) \neq 1$, $(3, a) = 1$, $n = 2\,a$ and that $d(m, n) = 2$ if m is odd and the same conditions hold. This proves part of the first and second case of (IV).

If $(7, a) = 1$, then $\#\,[a\,q\,(m + a\,q + a)] \mod 14 = \#\,[q\,(q + 1 + a^{-1}\,m)] \mod 14$. Hence if m is even, then $1 + a^{-1}\,m$ is odd modulo 14 and we calculate the number $q\,(q + 1)$ modulo 14 to obtain four different values. If m is odd, then $1 + a^{-1}\,m$ is even modulo 14 and we obtain the number of squares modulo 14, namely eight different values. This completes part of (I).

The case of $m' = 2\,q\,a - m$ reduces to exactly the same calculation as

$$(11) \qquad \mu(M_{m,2a}) - \mu(M_{2qa-m,2a}) = \frac{1}{14}\,a\,(1+q)\,(m - a\,q)$$

and we can get from (10) to (11) by the translation $q \mapsto -(1+q)$.

For $r = 0$ the conditions become $m \equiv \pm\, m' \mod a$ and

$$(12) \qquad \mu(M_{m,a}) - \mu(M_{qa+m,a}) = -\frac{1}{56}\,a\,q\,(2\,m + a\,q + a)$$

If $(7, a) \neq 1$, then $a = 7\,b$ and $\mu(M_{m,a}) - \mu(M_{qa+m,a}) = -\frac{1}{8}\,q\,(2\,m + 7\,b\,q + 7\,b)$. Therefore $\#\,[q\,(2\,m + 7\,b\,q + 7\,b)] \mod 8 = \#\,[q\,(q + 1 + 6\,b^{-1}\,m)] \mod 8 = \#\,[q\,(q + 1)] \mod 8$ as $(1 + 6\,b^{-1}\,m)$ is odd modulo 8. Hence we obtain four different values.

If $(7, a) = 1$, then $\#\,[a\,q\,(2\,m + a\,q + a)] \mod 56 = \#\,[q\,(q + 1 + 2\,a^{-1}\,m)] \mod 56 = \#\,[q\,(q + 1)] \mod 56$ and we obtain 16 different values.

The case of $m' = q\,a - m$ produces the same values as

$$(13) \qquad \mu(M_{m,a}) - \mu(M_{qa-m,a}) = \frac{1}{56}\,a\,(1+q)\,(2\,m - a\,q)$$

and we can get from (12) to (13) by the translation $q \mapsto -(1+q)$.

(II) $(3, a) \neq 1$

For $r \geq 4$ the conditions become $m \equiv \pm\, m' \mod 2^{r-2}a$ and we set $m' = q\,a\,2^{r-2} \pm m$ for some $q \in \mathbf{Z}$. Hence this case reduces to exactly the same calculation as for $(3, a) = 1$ and $r \geq 4$.

For $r = 3$ we obtain from the homotopy equivalence and homeomorphism conditions that $m \equiv \pm\, m' \mod 4a$ and we set $m' = 4\,q\,a \pm m$. The calculations are exactly the same as for $(3, a) = 1$ and $r = 3$.

For $r = 2$ the conditions reduce to $m \equiv \pm m' \mod 2a$, i.e. we first set $m' = 2\,q\,a + m$ and

$$(14) \qquad \mu(M_{m,4a}) - \mu(M_{2qa+m,4a}) = -\frac{1}{14}\,a\,q\,(m + a\,q + 2\,a)$$

If $(7, a) \neq 1$, then $a = 7\,b$ for some $b \in \mathbf{Z}$ and $\mu(M_{m,4a}) - \mu(M_{2qa+m,4a}) = -\frac{1}{2}\,b\,q\,(m + 7\,b\,q + 14\,b)$. Therefore $\#\,[b\,q\,(m + 7\,b\,q + 14\,b)] \mod 2 = \#\,[q\,(b\,q + m)]$ $\mod 2 = \#\,[q\,(q + b^{-1}\,m)] \mod 2 = \#\,[q\,(q + m)] \mod 2$. If m is even, we obtain the number of squares modulo 2, i.e. two different values. If m is odd, the calculation reduces to counting the number of consecutive products $q\,(q + 1)$ modulo 2 and we obtain one value.

If $(7, a) = 1$, then $\#\,[a\,q\,(m + a\,q + 2\,a)] \mod 14 = \#\,[q\,(q + 2 + a^{-1}\,m)]$ $\mod 14$. If m is even, then $2 + a^{-1}\,m$ is even modulo 14 and we get back to the number of squares modulo 14, i.e. we obtain eight different values. If m is odd, then $2 + a^{-1}\,m$ is odd and the number of $q\,(q + 1)$ modulo 14 leads to 4 different values.

For the case of $m' = 2\,q\,a - m$ we obtain the same values as

$$(15) \qquad \mu(M_{m,4a}) - \mu(M_{2qa-m,4a}) = \frac{1}{14}\,a\,(2 + q)\,(m - a\,q)$$

and we can get from (14) to (15) by the translation $q \mapsto -(2 + q)$

For $r = 1$ the conditions yield $m \equiv \pm m' \mod 2a$ and we get back to exactly the same computations as for $(3, a) = 1$ and $r = 1$.

For $r = 0$ the conditions yield $m \equiv \pm m' \mod a$ and we obtain exactly the same calculations as for $(3, a) = 1$ and $r = 0$. $\qquad\qquad\square$

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